Technical Results on Weak Bialgebras

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In these notes[2] I have gathered some technical results about (coquasi-triangular) weak bialgebras that I use in my research. Indeed even though [BNS99], [BS00] and [Nil] are the basic references on the topic, there is at the moment no single reference gathering all major results about this topic.

In Section 1 I give the definitions I use then in Section 2 I prove some properties of (coquasit-triangular) weak bialgebras.

1 Coquasi-Triangular Weak Hopf Algebras

Definition 1.1. A *weak bialgebra* $(H, \mu, \eta, \Delta, \varepsilon)$ over a field k is a vector space H such that

1. (H, μ, η) forms an associative algebra with multiplication $\mu : H \otimes H \to H$ and unit $\eta : k \to H$,
2. (H, Δ, ε) forms a coassociative coalgebra with comultiplication $\Delta : H \to H \otimes H$ and counit $\varepsilon : H \to k$,
3. the following compatibility conditions hold :
   - Multiplicativity of the Comultiplication :
   $$\Delta \circ \mu = (\mu \otimes \mu) \circ (\mathrm{id}_H \otimes \sigma_{H,H} \otimes \mathrm{id}_H) \circ (\Delta \otimes \Delta), \tag{1}$$
   - Weak Multiplicativity of the Counit :
   $$\begin{aligned} \varepsilon \circ \mu \circ (\mu \otimes \mathrm{id}_H) &= (\varepsilon \otimes \varepsilon) \circ (\mu \otimes \mu) \circ (\mathrm{id}_H \otimes \Delta \otimes \mathrm{id}_H) \\ &= (\varepsilon \otimes \varepsilon) \circ (\mu \otimes \mu) \circ (\mathrm{id}_H \otimes \Delta^{op} \otimes \mathrm{id}_H), \end{aligned} \tag{2}$$

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[2]Version of November 15, 2012

- Weak Comultiplicativity of the Unit :

$$\begin{aligned}(\Delta \otimes id_H) \circ \Delta \circ \eta &= (\mathrm{id}_H \otimes \mu \otimes \mathrm{id}_H) \circ (\Delta \otimes \Delta) \circ (\eta \otimes \eta) \\ &= (\mathrm{id}_H \otimes \mu^{op} \otimes \mathrm{id}_H) \circ (\Delta \otimes \Delta) \circ (\eta \otimes \eta), \quad (3)\end{aligned}$$

where $\sigma_{V,W} : V \otimes W \to W \otimes V : v \otimes w \mapsto w \otimes v$ flips the two tensor factors. Moreover $\mu^{op} = \mu \circ \sigma_{H,H}$ is the opposite multiplication and $\Delta^{op} = \sigma_{H,H} \circ \Delta$ is the opposite comultiplication. We also implicitly use Mac Lane's coherence theorem for the monoidal category **Vect** [Mac71, Chap. VII], identifying $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$ as well as $V \otimes k \cong V \cong k \otimes V$.

A *homomorphism* of weak bialgebras $\varphi : H \to H'$ is a homomorphism of both unital algebra and counital coalgebra.

Remark 1.2. The name weak bialgebra is fairly self explanatory. In particular, we see that it is the compatibility between the algebra and coalgebra structures that is weakened. In contrast to a bialgebra, the multiplicativity of the counit

$$\varepsilon \circ \mu = \varepsilon \otimes \varepsilon$$

and and the comultiplicativity of the unit

$$\Delta \circ \eta = \eta \otimes \eta$$

do not hold in general anymore and are replaced by (2) and (3) respectively. Also, the condition $\varepsilon \circ \eta = 1_k$ is absent.
From the above, we see that a weak bialgebra is a bialgebra if and only if we have

$$\varepsilon \circ \mu = \varepsilon \otimes \varepsilon, \ \Delta \circ \eta = \eta \otimes \eta, \ \varepsilon_s = \eta \circ \varepsilon, \text{ and } \varepsilon_t = \eta \circ \varepsilon.$$

Remark 1.3. Note that if H is a finite-dimensional weak bialgebra then so it H^*. We say that the definition is "self-dual".

Definition 1.4. Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a weak bialgebra over a field k. It is called *coquasi-triangular* if there exists a linear form $r : H \otimes H \to k$ called the *universal* r*-form*, that satisfies the following conditions :

i) For all $x, y \in H$

$$r(x \otimes y) = \varepsilon(x'y')r(x'' \otimes y'') = r(x' \otimes y')\varepsilon(y''x''). \quad (4)$$

ii) The form r has a weak convolution inverse, i.e. there exists $r^{-1} : H \otimes H \to k$ such that

$$r(x' \otimes y')\, r^{-1}(x'' \otimes y'') = \varepsilon(xy). \tag{5}$$
$$r^{-1}(x' \otimes y')\, r(x'' \otimes y'') = \varepsilon(yx), \tag{6}$$

iii) For all $x, y, z \in H$, we have

$$r(x' \otimes y')y''x'' = x'y'r(x'' \otimes y''), \tag{7}$$
$$r(xy \otimes z) = r(y \otimes z')r(x \otimes z''), \tag{8}$$
$$r(x \otimes yz) = r(x' \otimes y)r(x'' \otimes z). \tag{9}$$

Note that condition (7) implies that the commutativity inside H is "controlled" by the r-form, this why one often says that H is *almost commutative.*

A *homomorphism* of coquasi-triangular weak bialgebras $\varphi : (H, r) \to (H', r')$ is a homomorphism of weak bialgebra satisfying $r' \circ (\varphi \otimes \varphi) = r$.

Remark 1.5. A coquasi-triangular weak bialgebra that is a bialgebra is also coquasi-triangular as a bialgebra. In this case, one can simply omit (4) since it is automatically satisfied in a bialgebra. Moreover

$$r(x' \otimes y')\, r^{-1}(x'' \otimes y'') = \varepsilon(x)\varepsilon(y) = r^{-1}(x' \otimes y')\, r(x'' \otimes y'') \tag{10}$$

since in a bialgebra $\varepsilon(xy) = \varepsilon(x)\varepsilon(y) = \varepsilon(y)\varepsilon(x) = \varepsilon(yx)$.

Lemma 1.6. Let (H, r) be a coquasi-triangular weak bialgebra, then the coopposite weak bialgebra (H^{cop}, r^{-1}) is coquasi-triangular as well.

Remark 1.7. If we refer to "(8)" in the following, this indicates either the direct use of this equality for (H, r) or the use of the corresponding equality $r^{-1}(xy \otimes z) = r^{-1}(x \otimes z')r^{-1}(y \otimes z'')$ for (H^{cop}, r^{-1}). The context will every time make clear in which situation we are.

Definition 1.8. The weak bialgebra homomorphism

$$\varepsilon_t := (\varepsilon \otimes \mathrm{id}_H) \circ (\mu \otimes \mathrm{id}_H) \circ (\mathrm{id}_H \otimes \sigma_{H,H}) \circ (\Delta \otimes \mathrm{id}_H) \circ (\eta \otimes \mathrm{id}_H) \tag{11}$$

is called the *target counital map* whereas

$$\varepsilon_s = (\mathrm{id}_H \otimes \varepsilon) \circ (\mathrm{id}_H \otimes \mu) \circ (\sigma_{H,H} \otimes \mathrm{id}_H) \circ (\mathrm{id}_H \otimes \Delta) \circ (\mathrm{id}_H \otimes \eta) \tag{12}$$

is called the *source counital map.*

Definition 1.9. A *weak Hopf algebra* $(H, \mu, \eta, \Delta, \varepsilon, S)$ is a weak bialgebra $(H, \mu, \eta, \Delta, \varepsilon)$ with a linear map $S : H \to H$, called the *antipode*, that satisfies :

$$\mu \circ (S \otimes \mathrm{id}_H) \circ \Delta = \varepsilon_s, \tag{13}$$

$$\mu \circ (\mathrm{id}_H \otimes S) \circ \Delta = \varepsilon_t, \tag{14}$$

$$S = \mu \circ (\mu \otimes \mathrm{id}_H) \circ (S \otimes \mathrm{id}_H \otimes S) \circ (\Delta \otimes \mathrm{id}_H) \circ \Delta. \tag{15}$$

A *homomorphism* of weak Hopf algebras $\varphi : H \to H'$ is a homomorphism of weak bialgebras.

Remark 1.10. In this case, the Hopf algebra axioms

$$\mu \circ (S \otimes \mathrm{id}_H) \circ \Delta = \varepsilon \circ \eta \quad \text{and} \quad \mu \circ (\mathrm{id}_H \otimes S) \circ \Delta = \varepsilon \circ \eta$$

are weakened to (13) and (14) respectively whereas (15) is new.

Example 1.11. Let k be a field, $G = (G_0, G_1)$ be a groupoid and $f, f' \in G_1$. Then groupoid algebra $k[G]$ has a weak Hopf algebra structure given by

$$\begin{aligned}
\mu(f \otimes f') &= \begin{cases} f \circ f' & \text{if target(f') = source(f)} \\ 0 & \text{otherwise} \end{cases}, \\
\eta(1) &= \sum_{x \in G_0} \mathrm{id}_x, \\
\Delta(f) &= f \otimes f, \\
\varepsilon(f) &= 1 \qquad \forall f \in G_1, \\
S(f) &= f^{-1}.
\end{aligned}$$

Note that, due to its construction, a groupoid algebra is always cocommutative. Moreover, this is an example of the weak Hopf algebra that is *not* a Hopf algebra.

Lemma 1.12. Let $\varphi : H \to H'$ be a homomorphism of weak bialgebras and let H, H' be weak Hopf algebras. Then $S' \circ \varphi = \varphi \circ S$.

Proof. Using the weak Hopf algebra axioms, we find

$$\begin{aligned}
S'(\varphi(x)) &\overset{15}{=} S'(\varphi(x)')\varphi(x)''S'(\varphi(x)''') \\
&\overset{13}{=} \varepsilon_s'(\varphi(x)')S'(\varphi(x)'') \\
&= \varepsilon_s'(\varphi(x'))S'(\varphi(x'')) \\
&\overset{\star}{=} \varphi(\varepsilon_s(x'))S'(\varphi(x'')) \\
&\overset{13}{=} \varphi(S(x')x'')S'(\varphi(x''')) \\
&= \varphi(S(x'))\varphi(x'')S'(\varphi(x''')) \\
&= \varphi(S(x'))\varphi(x'')'S'(\varphi(x'')'') \\
&\overset{14}{=} \varphi(S(x'))\varepsilon_t'(\varphi(x'')) \\
&\overset{\star}{=} \varphi(S(x'))\varphi(\varepsilon_t(x'')) \\
&\overset{14}{=} \varphi(S(x')a''S(x''')) \\
&\overset{15}{=} \varphi(S(x)),
\end{aligned}$$

where $\star$ uses that $\varphi(1) = 1'$ and $\varepsilon'(\varphi(x)) = \varepsilon(x)$ and thus $\varphi(\varepsilon_s(x)) = \varepsilon_s'(x)$ and $\varphi(\varepsilon_t(x)) = \varepsilon_t'(x)$. □

Notation 1.13. From now on we shall abbreviate weak bialgebra by WBA and weak Hopf algebra by WHA. Moreover, coquasi-triangular will be written "CQT"; thus a coquasi-triangular weak bialgebra will thus be called CQT WBA.

We shall now introduce a concept that will play an important role in the rest of this thesis.

Definition 1.14. Let H be a WBA. An element $g \in H$ is called *right group-like* if

$$\Delta(g) = g1' \otimes g1'' \qquad \text{and} \qquad \varepsilon_s(g) = 1, \tag{16}$$

it is called *left group-like* if

$$\Delta(g) = 1'g \otimes 1''g \qquad \text{and} \qquad \varepsilon_t(g) = 1. \tag{17}$$

An element $g \in H$ is called *group-like* if it is both right and left group-like. We denote the set of group-like elements of H by $G(H)$.

Notation 1.15. In what follows we sometimes have two or more units showing up in our computations. In order differentiate them and keep track of which one is which one, we use subscripts. Hence we have, for example, $\varepsilon_s(a) = 1'\varepsilon(a1'')$ and then $1\varepsilon_s(a) = 1_1 1_2'\varepsilon(a1_2'')$.

Lemma 1.16. The set of group-like elements $G(H)$ of a WBA H is a monoid.

Proof. i) $1 \in H$ is group-like.
We have $1_1 \cdot 1'_2 \otimes 1_3 \cdot 1''_2 = 1' \otimes 1'' = \Delta(1)$ and similarly $1'_1 \cdot 1_2 \otimes 1''_1 \cdot 1_3 = \Delta(1)$. Furthermore, $\varepsilon_s(1) = 1'_1 \varepsilon(1_2 \cdot 1''_1) = 1$ and $\varepsilon_t(1) = \varepsilon(1'_1 \cdot 1_2) 1''_1 = 1$.

ii) If $g, h \in G(H)$ then $gh \in G(H)$.
We have

$$\begin{aligned}\Delta(gh) &= (gh)' \otimes (gh)'' = g'h' \otimes g''h'' = (g1'_1)(1'_2 h) \otimes (g1''_1)(1''_2 h) \\ &= g(1'_1 1'_2 h) \otimes g(1''_1 1''_2 h) = g(1'h) \otimes g(1''h) = g(h1') \otimes g(h1'') \\ &= (gh)1' \otimes (gh)1'',\end{aligned}$$

by definition of the comultiplication; associativity; definition of group-like; associativity; associativity and unit axiom; definition of group-like; associativity.
Similarly, $\Delta(gh) = 1'(gh) \otimes 1''(gh)$. We furthermore have

$$\varepsilon_s(gh) \overset{20}{=} \varepsilon_s(\varepsilon_s(g)h) = \varepsilon(1 \cdot h) = \varepsilon(h) = 1,$$

where we have used that g is group-like. In a similar way, $\varepsilon_t(gh) = 1$.
□

Lemma 1.17. Let H be a WHA. Then every group-like is invertible with $g^{-1} = S(g)$ and $G(H)$ forms a group.

Proof. Let $g \in H$ be group-like. Then

$$\begin{aligned}S(g)g &= S(g)1g = S(g)\varepsilon_s(1)g = S(g)s(1')1''g \\ &= S(1'g)1''g = S(g')g'' = \varepsilon_s(g) = 1,\end{aligned}$$

and similarly $gS(g) = 1$. Hence $g^{-1} = S(g)$.

Let us now look at the group structure of $G(H)$. From the previous lemma we know that 1 is group-like and that if g and h are in $G(H)$ then so is gh. It remains to prove that for g group-like, so is g^{-1}. We have

$$\begin{aligned}\Delta(g^{-1}) &= (g^{-1})' \otimes (g^{-1})'' = g^{-1}g(g^{-1})' \otimes g^{-1}g(g^{-1})'' \\ &= g^{-1}g(1g^{-1})' \otimes g^{-1}g(1g^{-1})'' = g^{-1}g1'(g^{-1})' \otimes g^{-1}g1''(g^{-1})'' \\ &= g^{-1}(g1')(g^{-1})' \otimes g^{-1}(g1'')(g^{-1})'' = g^{-1}g'(g^{-1})' \otimes g^{-1}g''(g^{-1})'' \\ &= g^{-1}(gg^{-1})' \otimes g^{-1}(gg^{-1})'' \\ &= g^{-1}1' \otimes g^{-1}1'',\end{aligned}$$

and similarly $\Delta(g^{-1}) = 1'g^{-1} \otimes 1''g^{-1}$. Finally, we have

$$\varepsilon_s(g^{-1}) = \varepsilon_s(1g^{-1}) = \varepsilon_s(\varepsilon_s(g)g^{-1}) \overset{oo}{=} \varepsilon_s(gg^{-1}) = \varepsilon_s(1) = 1,$$

and similarly $\varepsilon_t(g^{-1}) = 1$, hence g^{-1} is group-like.

□

Convention 1.18. In what follows we shall abbreviate weak bialgebra by "WBA" and coquasi-triangular by "CQT".

2 Technical Results about WBAs

In this section we present technical results need in the rest of this thesis. Most of the results presented here are scattered around the literature while others are commonly used but not proved in any paper. Hence, out of completeness, we prove here (nearly all) the lemmata and propositions we shall need in the next chapters.

Lemma 2.1. Let H be a WBA, $x, y \in H$. We have

$$\varepsilon_s(1') \otimes 1'' = 1' \otimes 1'' \quad \text{and} \quad 1' \otimes \varepsilon_t(1'') = 1' \otimes 1'', \tag{18}$$

$$\varepsilon(\varepsilon_s(x)y) = \varepsilon(xy) \quad \text{and} \quad \varepsilon(x\varepsilon_t(y)) = \varepsilon(xy), \tag{19}$$

$$\varepsilon_s(\varepsilon_s(x)y) = \varepsilon_s(xy) \quad \text{and} \quad \varepsilon_t(x\varepsilon_t(y)) = \varepsilon_t(xy). \tag{20}$$

Proof. i) We have $\varepsilon_s(1') \otimes 1'' = 1'_1\varepsilon(1'_2 1''_1) \otimes 1''_2 \overset{3}{=} 1'\varepsilon(1'') \otimes 1''' = 1' \otimes 1''$, and similarly $1' \otimes \varepsilon_t(1'') = 1' \otimes 1''$.

ii) We have $\varepsilon(\varepsilon_s(x)y) = \varepsilon(1'\varepsilon(x1'')y) = \varepsilon(1'y)\varepsilon(x1'') \overset{2}{=} \varepsilon(x1y) = \varepsilon(xy)$. We similarly prove that $\varepsilon(x\varepsilon_t(y)) = \varepsilon(xy)$.

iii) Equalities 20 are direct consequences of 18 and 2.

□

Lemma 2.2. Let H be a WBA and $x \in H$. Then

$$\varepsilon_s(x) = 1'\varepsilon(\varepsilon_s(x)1''), \tag{21}$$

$$\Delta\varepsilon_s(x) = 1' \otimes \varepsilon_s(x)1'', \tag{22}$$

$$1' \otimes 1''\varepsilon_s(x) = \varepsilon_s(x)' \otimes \varepsilon_s(x)'', \tag{23}$$

$$x' \otimes \varepsilon_s(x'') = x1'' \otimes \varepsilon_s(1''), \tag{24}$$

$$\varepsilon_s(a)' \otimes \varepsilon_s(a)'' = \varepsilon_s(\varepsilon_s(a)') \otimes \varepsilon_s(a)'', \tag{25}$$

$$\varepsilon_t(a)' \otimes \varepsilon_t(a)'' = \varepsilon_t(a)' \otimes \varepsilon_t(\varepsilon_t(a)''). \tag{26}$$

Proof. i) Using (2), we have

$$\begin{aligned}1'\varepsilon(\varepsilon_s(x)1'') &= 1'_2\varepsilon(1'_1\varepsilon(x1''_1)1''_2)\\ &= 1'_2\varepsilon(x1''_1)\varepsilon(1'_11''_2)\\ &= \varepsilon(x1_11''_2)1'_2\\ &= 1'\varepsilon(x1'')\\ &= \varepsilon_s(x).\end{aligned}$$

ii) Using (3) we find

$$\begin{aligned}\Delta\varepsilon_s(x) &= \Delta(1'\varepsilon(x1''))\\ &= 1'\otimes 1''\varepsilon(x1''')\\ &= 1'_1\otimes 1'_21''_1\varepsilon(x1''_2)\\ &= 1'\otimes\varepsilon_s(x)1''.\end{aligned}$$

iii) Using (3) we find

$$\begin{aligned}1'\otimes 1''\varepsilon_s(x) &= 1'_1\otimes 1''_11'_2\varepsilon(x1''_2)\\ &= 1'\otimes 1''\varepsilon(x1''')\\ &= (1')'\otimes(1')''\varepsilon(x1'')\\ &= \varepsilon_s(x)'\otimes\varepsilon_s(x)''.\end{aligned}$$

iv) We have

$$\begin{aligned}x'\otimes\varepsilon_s(x'') &= x'\otimes 1'\varepsilon(x''1'')\\ &= (x1_1)'\varepsilon((x1_1)''1''_2)\otimes 1'_2\\ &= x1'_1\varepsilon(x''1''_11''_2)\otimes 1'_2\\ &\stackrel{2}{=} x'1'_1\varepsilon(x''1''_1)\varepsilon(1'''_11''_2)\otimes 1'_1\\ &= x(1'_1)'\varepsilon(x''(1'_1)'')\varepsilon(1''_11''_2)\otimes 1'_2\\ &= (x1'_1)'\varepsilon((x1_1)'')\varepsilon(1''_11''_2)\otimes 1'_2\\ &= x1_1\otimes 1'_2\varepsilon(1''_11''_2)\\ &= x1'\otimes\varepsilon_s(1'').\end{aligned}$$

v) Using (3) again, we find

$$\begin{aligned}\varepsilon_s(a)'\otimes\varepsilon_s(a)'' &= 1'\otimes 1''\varepsilon(a1''') = 1'_1\otimes 1''_1 1'_2\varepsilon(a1''_2)\\ &= 1'_1\varepsilon(1''_1)\otimes 1'''_1 1'_2\varepsilon(a1''_2) = 1'_1\varepsilon(1'_2 1''_1)\otimes 1''_2 1'_3\varepsilon(a1''_3)\\ &= \varepsilon_s(1'_1)\otimes 1''_1 1'_2\varepsilon(a1''_2) = \varepsilon_s(1')\otimes 1''\varepsilon(a1''')\\ &= \varepsilon_s(\varepsilon_s(a)')\otimes\varepsilon_s(a)''.\end{aligned}$$

We prove in a similar way that $\varepsilon_t(a)'\otimes\varepsilon_t(a)'' = \varepsilon_t(a)'\otimes\varepsilon_t(\varepsilon_t(a)'')$.

□

Lemma 2.3. Let (H, r) be a CQT WBA. Then

$$r(a\otimes 1) = r(1\otimes a) = \varepsilon(a). \tag{27}$$

Proof. We have

$$\begin{aligned}\varepsilon(a) &= \varepsilon(1\cdot a) \overset{5}{=} r(1'\otimes a')r^{-1}(1''\otimes a'') = r(1'_1\cdot 1_2\otimes a')r^{-1}(1''_1\otimes a'')\\ &\overset{8}{=} r(1_2\otimes a')r(1'_1\otimes a'')r^{-1}(1''_1\otimes a''') \overset{5}{=} r(1_2\otimes a')\varepsilon(1_1\cdot a'')\\ &= r(1\otimes a).\end{aligned}$$

Similarly, $\varepsilon(a) = r(a\otimes 1)$. □

The next lemma will give us the tools required to prove (41).

Lemma 2.4. Let H be a WBA and $a, b\in H$. Then

$$\varepsilon_t(a')\otimes\varepsilon_s(a'') = \varepsilon_t(a'')\otimes\varepsilon_s(a'), \tag{28}$$

$$\varepsilon_t(a')\varepsilon(a''b) = \varepsilon_t(ab), \tag{29}$$

$$\varepsilon(ab')\varepsilon_s(b'') = \varepsilon_s(ab), \tag{30}$$

$$\varepsilon_t((ab)')\otimes\varepsilon_s((ab)'') = \varepsilon_t(a')\varepsilon(a''b')\otimes\varepsilon_s(b''), \tag{31}$$

$$\varepsilon_t((ab)')\otimes\varepsilon_s((ab)'') = \varepsilon_t(a')\otimes\varepsilon_s(b')\varepsilon(a''b''), \tag{32}$$

$$\varepsilon_t((ab)')\otimes\varepsilon_s((ab)'') = \varepsilon(a'b')\varepsilon_t(a'')\otimes\varepsilon_s(b''). \tag{33}$$

Proof. i) We have

$$\begin{aligned}\varepsilon_t(a')\otimes\varepsilon_s(a'') &= \varepsilon(1'_1a')1''_1\otimes 1'_2\varepsilon(a''1''_2)\\ &\overset{2}{=} \varepsilon(1'_1a'')1''_1\otimes 1'_2\varepsilon(a'1''_2)\\ &= \varepsilon_t(a'')\otimes\varepsilon_s(a').\end{aligned}$$

ii) Here we have

$$\varepsilon_t(a')\varepsilon(a''b) = \varepsilon(1'a')1''\varepsilon(a''b) \overset{2}{=} \varepsilon(1'ab)1'' = \varepsilon_t(ab).$$

We similarly prove that $\varepsilon(ab')\varepsilon_s(b'') = \varepsilon_s(ab)$.

iii) We have

$$\begin{aligned}\varepsilon_t((ab)') \otimes \varepsilon_s((ab)'') &= \varepsilon_t(a'b') \otimes \varepsilon_s(a''b'') \\ &\overset{\diamondsuit}{=} \varepsilon_t(a')\varepsilon(a''b') \otimes \varepsilon(a'''b'')\varepsilon_s(b''') \\ &= \varepsilon_t(a')\varepsilon(a''b') \otimes \varepsilon_s(b''),\end{aligned}$$

where $\diamondsuit$ follows from (29) and (30).

iv) We have

$$\begin{aligned}\varepsilon_t((ab)') \otimes \varepsilon_s((ab)'') &\overset{31}{=} \varepsilon_t(a')\varepsilon(a''b') \otimes \varepsilon_s(b'') \\ &\overset{19}{=} \varepsilon_t(a')\varepsilon(a''\varepsilon_t(b')) \otimes \varepsilon_s(b'') \\ &\overset{28}{=} \varepsilon_t(a')\varepsilon(a''\varepsilon_t(b'')) \otimes \varepsilon_s(b') \\ &\overset{19}{=} \varepsilon_t(a') \otimes \varepsilon_s(b')\varepsilon(a''b'').\end{aligned}$$

We similarly prove that $\varepsilon_t((ab)') \otimes \varepsilon_s((ab)'') = \varepsilon(a'b')\varepsilon_t(a'') \otimes \varepsilon_s(b'')$.

□

The next lemma will help us prove (44).

Lemma 2.5. Let H be a WBA and $a, b \in H$. Then

$$a\varepsilon_t(b) = \varepsilon(a'b)a'', \tag{34}$$

$$\varepsilon_s(a)b = b'\varepsilon(ab''), \tag{35}$$

$$\varepsilon(ac')\varepsilon(bc'') = \varepsilon(a\varepsilon_t(c)')\varepsilon(b\varepsilon_t(c)''), \tag{36}$$

$$\varepsilon_t(a') \otimes \varepsilon_t(a'') = \varepsilon_t(\varepsilon_t(a)') \otimes \varepsilon_t(\varepsilon_t(a)''), \tag{37}$$

$$\varepsilon(ab')\varepsilon_s(\varepsilon_t(b'')) = \varepsilon(a'b)\varepsilon_s(a''). \tag{38}$$

(39)

If H is moreover CQT with r-form r, then

$$r(a' \otimes b)\varepsilon_s(\varepsilon_t(a'')) = r(a \otimes b')\varepsilon_s(b''). \tag{40}$$

Proof. i) We have

$$\begin{aligned} a\varepsilon_t(b) &= \varepsilon((a\varepsilon_t(b))')(a\varepsilon_t(b))'' \\ &= \varepsilon((a\varepsilon(1'b)1'')')(a\varepsilon(1'b)1'')'' \\ &= \varepsilon(a'1'')\varepsilon(1'b)a''1''' \\ &\overset{2}{=} \varepsilon(a'1'b)a''1'' \\ &= \varepsilon((a1)'b)(a1)'' \\ &= \varepsilon(a'b)a''. \end{aligned}$$

We similarly prove that $\varepsilon_s(a)b = b'\varepsilon(ab'')$.

ii) We have

$$\varepsilon(ac')\varepsilon(bc'') \overset{35}{=} \varepsilon(a\varepsilon_s(b)c) \overset{19}{=} \varepsilon(a\varepsilon_s(b)\varepsilon_s(c)) \overset{35}{=} \varepsilon(a\varepsilon_t(c)')\varepsilon(b\varepsilon_t(c)'').$$

iii) We have

$$\begin{aligned} \varepsilon_t(a') \otimes \varepsilon_t(a'') &= \varepsilon(1'_1a')1''_1 \otimes \varepsilon(1'_2a'')1''_2 \\ &\overset{36}{=} \varepsilon(1'_1\varepsilon_t(a)')1''_1 \otimes \varepsilon(1'_2\varepsilon_t(a)'')1''_2 \\ &= \varepsilon_t(\varepsilon_t(a)') \otimes \varepsilon_t(\varepsilon_t(a)''). \end{aligned}$$

iv) We have

$$\begin{aligned} \varepsilon(ab')\varepsilon_s(\varepsilon_t(b'')) &\overset{19}{=} \varepsilon(a\varepsilon_t(b'))\varepsilon_s(\varepsilon_t(b'')) \\ &\overset{37}{=} \varepsilon(a\varepsilon_t(\varepsilon_t(b)'))\varepsilon_s(\varepsilon_t(\varepsilon_t(b)'')) \\ &\overset{\diamondsuit}{=} \varepsilon(a\varepsilon_t(b)')\varepsilon_s(\varepsilon_t(b)'') \\ &\overset{30}{=} \varepsilon_s(a\varepsilon_t(b)) \\ &\overset{34}{=} \varepsilon(a'b)\varepsilon_s(b''), \end{aligned}$$

where $\diamondsuit$ follows from (19) and (26).

v) We have

$$\begin{aligned} r(a' \otimes b)\varepsilon_s(\varepsilon_t(a'')) &\overset{4}{=} r(a' \otimes b')\varepsilon(b''a'')\varepsilon_s(\varepsilon_t(a''')) \\ &\overset{38}{=} r(a' \otimes b')\varepsilon(b''a'')\varepsilon_s(b''') \\ &\overset{4}{=} r(a \otimes b')\varepsilon_s(b''). \end{aligned}$$

□

Lemma 2.6. Let (H, r) be a CQT WBA and $a, b \in H$. Then

$$\varepsilon_t(a') \otimes \varepsilon_s(b')r(a'' \otimes b'') = r(a' \otimes b')\varepsilon_t(b'') \otimes \varepsilon_s(a''), \tag{41}$$
$$r(a \otimes \varepsilon_t(b)) = \varepsilon(ab), \tag{42}$$
$$r(a \otimes \varepsilon_s(b)) = \varepsilon(ba), \tag{43}$$
$$r(\varepsilon_s(a) \otimes b) = \varepsilon(b\varepsilon_s(a)), \tag{44}$$
$$r^{-1}(a \otimes \varepsilon_s(b)) = \varepsilon(a\varepsilon_s(b)), \tag{45}$$
$$r^{-1}(\varepsilon_t(a) \otimes b) = \varepsilon(ba). \tag{46}$$

Proof. i) We have

$$\begin{aligned}
\varepsilon_t(a') \otimes \varepsilon_s(b')r(a'' \otimes b'') &\overset{4}{=} \varepsilon_t(a') \otimes \varepsilon_s(b')\varepsilon(a''b'')r(a''' \otimes b''') \\
&\overset{32}{=} \varepsilon_t((a'b')') \otimes \varepsilon_s((a'b')'')r(a'' \otimes b'') \\
&\overset{7}{=} r(a' \otimes b')\varepsilon_t((b'a')') \otimes \varepsilon_s((b'a')'') \\
&\overset{33}{=} r(a' \otimes b')\varepsilon(b''a'')\varepsilon_t(b''') \otimes \varepsilon_s(a''') \\
&\overset{4}{=} r(a' \otimes b')\varepsilon_t(b'') \otimes \varepsilon_s(a'').
\end{aligned}$$

ii) We have

$$\begin{aligned}
r(a \otimes \varepsilon_t(b)) &= r(a \otimes \varepsilon(1'b)1'') = \varepsilon(1'b)r(a \otimes 1'') \\
&\overset{19}{=} \varepsilon(\varepsilon_s(1')b)r(a \otimes 1'') = \varepsilon(a')\varepsilon(\varepsilon_s(1')b)r(a'' \otimes 1'') \\
&= \varepsilon(\varepsilon_t(a'))\varepsilon(\varepsilon_s(1')b)r(a'' \otimes 1'') \\
&\overset{41}{=} r(a' \otimes 1')\varepsilon(\varepsilon_s(a'')b)\varepsilon(\varepsilon_t(1'')) \\
&= r(a' \otimes 1')\varepsilon(\varepsilon_s(a'')b)\varepsilon(1'') = r(a' \otimes 1)\varepsilon(\varepsilon_s(a'')b) \\
&= r(a' \otimes 1)\varepsilon(a''b) \overset{27}{=} \varepsilon(a')\varepsilon(a''b) \\
&= \varepsilon(ab).
\end{aligned}$$

Similarly, one has that $r(a \otimes \varepsilon_s(b)) = \varepsilon(ba)$.

iii) We have

$$\begin{aligned}
r(\varepsilon_s(a)\otimes b) &= r(1'\varepsilon(a1'')\otimes b)\\
&\overset{\diamondsuit}{=} \varepsilon(\varepsilon_s(a)\varepsilon_t(1''))r(1'\otimes b)\\
&= \varepsilon(1'_2\varepsilon(a1''_2)\varepsilon(1'_3 1''_1)1''_3)r(1'_1\otimes b)\\
&= \varepsilon(1'_3 1''_1)\varepsilon(1'_2 1''_3)\varepsilon(a1''_2)r(1'_1\otimes b)\\
&\overset{3}{=} \varepsilon(1'_3 1''_1)\varepsilon(1''_3 1'_2)\varepsilon(a1''_2)r(1'_1\otimes b)\\
&= \varepsilon(\varepsilon(1'_3 1''_1)1''_3 1'_2\varepsilon(a1''_2))r(1'_1\otimes b)\\
&= \varepsilon(\varepsilon_t(1'')\varepsilon_s(a))r(1'\otimes b)\\
&\overset{19}{=} \varepsilon(\varepsilon_s(\varepsilon_t(1''))\varepsilon_s(a))r(1'\otimes b)\\
&\overset{40}{=} \varepsilon(\varepsilon_s(b'')\varepsilon_s(a))r(1\otimes b')\\
&\overset{\star}{=} \varepsilon(b''\varepsilon_s(a))\varepsilon(b')\\
&= \varepsilon(b\varepsilon_s(a)),
\end{aligned}$$

where $\diamondsuit$ follows from (18) and (19) and $\star$ from (19) and (27).

□

Notation 2.7. We what follows we often take many times the comultiplication of an element. In order to make it easy to read we slightly modify Sweedler's notation; namely we use roman numbers to "orders" higher than three thus a'''' becomes a^{IV} whereas a''''' is written a^{V}. Using this convention we have, for example,

$$\begin{aligned}
(\Delta\otimes\Delta\otimes\Delta)\circ(\Delta\otimes\mathrm{id})\circ\Delta(a) &= (\Delta\otimes\Delta\otimes\Delta)(a'\otimes a''\otimes a''')\\
&= a'\otimes a''\otimes a'''\otimes a^{IV}\otimes a^{V}\otimes a^{VI}.
\end{aligned}$$

Proposition 2.8. Let (H,r) be a CQT WBA, $a,b\in H$ and $g\in G(H)$ a group-like element. Then

$$\text{i)}\quad r^{-1}(a'\otimes g)r(a''\otimes g)=\varepsilon(a), \tag{47}$$

$$\text{ii)}\quad r(a'\otimes g)r^{-1}(a''\otimes g)=\varepsilon(a), \tag{48}$$

$$\text{iii)}\quad r(a\otimes g')r(b\otimes g'')=r(a'\otimes g)r(b'\otimes g)\varepsilon(b''a''), \tag{49}$$

$$\text{iv)}\quad r^{-1}(a\otimes g')r^{-1}(b\otimes g)=\varepsilon(a'b')r^{-1}(a''\otimes g)r^{-1}(b''\otimes g). \tag{50}$$

Proof. In this proof, we indicate by $*$ the equalities where we use that g is group-like.

i) We have

$$\begin{aligned}
r^{-1}(a'\otimes g)r(a''\otimes g) &= r^{-1}(a'\otimes g)\varepsilon((a''1)')\varepsilon((a''1)'')r(a^{IV}\otimes g)\\
&= r^{-1}(a'\otimes g)\varepsilon(a''1')\varepsilon(a'''1'')r(a^{IV}\otimes g)\\
&\overset{42}{=} r^{-1}(a'\otimes g)\varepsilon(a''1')r(a'''\otimes\varepsilon_t(1''))r(a^{IV}\otimes g)\\
&\overset{18}{=} r^{-1}(a'\otimes g)\varepsilon(a''1')r(a'''\otimes 1'')r(a^{IV}\otimes g)\\
&\overset{9}{=} r^{-1}(a'\otimes g)\varepsilon(a''1')r(a'''\otimes 1''g)\\
&\overset{18}{=} r^{-1}(a'\otimes g)\varepsilon(a''\varepsilon_s(1'))r(a'''\otimes 1''g)\\
&\overset{45}{=} r^{-1}(a'\otimes g)r^{-1}(a''\otimes\varepsilon_s(1'))r(a'''\otimes 1''g)\\
&\overset{18}{=} r^{-1}(a'\otimes g)r^{-1}(a''\otimes 1')r(a'''\otimes 1''g)\\
&\overset{9}{=} r^{-1}(a'\otimes 1'g)r(a''\otimes 1''g)\\
&\overset{*}{=} r^{-1}(a'\otimes g')r(a''\otimes g'')\\
&\overset{6}{=} \varepsilon(ga)\overset{19}{=}\varepsilon(\varepsilon_s(g)a)\overset{*}{=}\varepsilon(1a)\\
&= \varepsilon(a).
\end{aligned}$$

ii) Similar to i).

iii) Here we have

$$\begin{aligned}
r(a\otimes g')r(b\otimes g'') &\overset{*}{=} r(a\otimes g1')r(b\otimes g1'')\\
&\overset{9}{=} r(a'\otimes g)r(a''\otimes 1')r(b'\otimes g)r(b''\otimes 1'')\\
&\overset{8}{=} r(a'\otimes g)r(b'\otimes g)r(b''a''\otimes 1)\\
&\overset{27}{=} r(a'\otimes g)r(b'\otimes g)\varepsilon(b''a'').
\end{aligned}$$

iv) Similar to iii).

□

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